Abstracts

Mirosław Adamek
(University of Bielsko-Biała)

On a generalization of sandwich type theorems

In this talk we introduce the affine and convex functions with a control function and present some sandwich type theorems for them. The presented results generalize the well known results for standard affine and convex functions and also strongly convex functions (cf. [1], [3], [2]).

References

[1] K. Baron, J. Matkowski, K. Nikodem, *A sandwich with convexity*, Math. Pannon. **5** (1994), 139–144.
[2] N. Merentes, K. Nikodem, *Strong convexity and separation theorems*, Aequationes Math. **90** (2016), 47–55.
[3] K. Nikodem, Sz. Wąsowicz, *A sandwich theorem and Hyers-Ulam stability of affine functions*, Aequationes Math. **49** (1995), 160–164.

Ádám Besenyei
(Eötvös Loránd University)

The irresistible inequality of Milne

We focus on a neat elementary inequality established by the British physicist Edward Arthur Milne in 1925 as a tool for a particular estimate related to astrophysical quantities. We first reveal the surprising hidden link between Milne's inequality and certain electric circuits consisting of resistors and switches. Then, as an application of the inequality, we provide an alternative proof for the strong subadditivity of entropy in the discrete case. Besides mathematics, we also discuss some interesting historical details.

Mihály Bessenyei
(University of Debrecen)

The contraction principle in extended context

There are several extensions of the classical Banach Fixed Point Theorem in technical literature. A branch of generalizations replaces usual contractivity by weaker but still effective assumptions. Our talk follows this stream, presenting an elementary proof for a known fixed point result. Some applications are also given.

Zoltán Boros
(University of Debrecen)

Schwarz inequality over groups

The following abstract versions of the celebrated Schwarz inequality are established:

Theorem. *Let $(X, +)$ denote a group and $F\colon X \times X \to \mathbb{R}$ be a symmetric, bi-additive mapping such that $F(x, x) \geq 0$ for every $x \in X$. Then*

$$|F(x, y)| \leq \sqrt{F(x, x)} \cdot \sqrt{F(y, y)}$$

holds for every $x, y \in X$.

Theorem. *Let $(X, +)$ denote an Abelian group and let us suppose that $F\colon X \times X \to \mathbb{R}$ satisfies the following conditions for every $u, x, y, z \in X$:*

(i) $F(x + z, y + u) + F(x - z, y - u) = 2F(x, y) + 2F(z, u)$ (F is quadratic),
(ii) $F(x, y) = F(y, x)$ (F is symmetric) and
(iii) $F(x, -x) \leq F(x, x)$.

Then

$$|F(x, y)| \leq |F(x, 0)| + |F(0, y)| + \frac{1}{2} \cdot \sqrt{F(x, x) - F(x, -x)} \cdot \sqrt{F(y, y) - F(y, -y)}$$

holds for every $x, y \in X$.

The first result was obtained in a joint work with Árpád Száz. The second statement combines the first one with a decomposition theorem for quadratic functions on product spaces.

Pál Burai
(University of Debrecen)

On certain Schur-convex functions

(joint work with Judit Makó)

Let $D \subset X$ be a convex, non-empty set, where X is a linear space, a function $f\colon D \times D \to \mathbb{R}$ is said to be Schur-convex, if

$$f(tx + (1-t)y, (1-t)x + ty) \leq f(x, y)$$

for all $x, y \in D$ and for all $t \in [0, 1]$. The main goal of this talk is the investigation of the properties of this class of functions.

Jacek Chmieliński

(Pedagogical University in Kraków)

Approximate Birkhoff orthogonality

(joint work with Paweł Wójcik)

Let $(X, \|\cdot\|)$ be a normed space over $\mathbb{K}$. For $x, y \in X$ and $\varepsilon \in [0,1)$, we follow [1] and define an ε-orthogonality:

$$x \perp_{\mathrm{B}}^{\varepsilon} y \quad :\Leftrightarrow \quad \forall\, \lambda \in \mathbb{K}: \quad \|x+\lambda y\|^2 \geq \|x\|^2 - 2\varepsilon \|x\|\,\|\lambda y\|.$$

Obviously, for $\varepsilon = 0$, we get the classical definition of the Birkhoff orthogonality:

$$x \perp_{\mathrm{B}} y \quad \Leftrightarrow \quad \forall\, \lambda \in \mathbb{K} \quad \|x+\lambda y\| \geq \|x\|.$$

Moreover, if X is an inner product space, we have

$$x \perp_{\mathrm{B}}^{\varepsilon} y \quad \Leftrightarrow \quad |\langle x|y\rangle| \leq \varepsilon \|x\|\,\|y\| \quad \Leftrightarrow: \quad x \perp^{\varepsilon} y.$$

In this talk, we present a new characterization of the ε-orthogonality:

$$x \perp_{\mathrm{B}}^{\varepsilon} y \quad \Leftrightarrow \quad \exists\, z \in X: \quad x \perp_{\mathrm{B}} z, \quad \|z - y\| \leq \varepsilon \|y\|$$

which, in particular, yields:

$$x \perp_{\mathrm{B}}^{\varepsilon} y \quad \Leftrightarrow \quad \exists\, \varphi \in X^*: \ \|\varphi\| = 1,\ \varphi(x) = \|x\|,\ |\varphi(y)| \leq \varepsilon \|y\|.$$

Some applications to operator theory will also be given.

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Jacek Chudziak
(University of Rzeszów)

Zero utility principle as a quasideviation mean

Quasideviation means, introduced in [6], are generalizations of the deviation means, investigated in [2]-[4]. A related concept is that of implicit means, studied in [5]. A series of properties of quasidevaition means have been proved in [7]. It turns out that an important notion of insurance mathematics, namely the zero utility principle introduced in [1], is a particular case of the quasideviation mean. Applying the results in [7], we obtain the results concerning comparison, equality, positive homogeneity, subadditivity and convexity of the zero utility principle.

References

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Włodzimierz Fechner
(University of Silesia)

On some functional inequalities for lattice-valued mappings

(joint work with Nutefe Kwami Agbeko and Ewa Rak)

Assume that $(G, *)$ is a semigroup, $(L, \leq)$ is a lattice and $T\colon G \to L$ is a lattice-valued mapping. Inspired by the notion of optimal average introduced by the first coauthor we ask about morphisms between the algebraic structure of G and the order structure of L. To be precise, we are interested in the following functional equation:

$$T(x * y) = T(x) \vee T(y), \quad x, y \in G \tag{1}$$

and in the two related functional inequalities:

$$T(x * y) \geq T(x) \vee T(y), \quad x, y \in G \tag{2}$$

and

$$T(x * y) \leq T(x) \vee T(y), \quad x, y \in G. \tag{3}$$

We also discuss stability properties of functional equation (1).

György Gát
(University of Debrecen)

Marcinkiewicz and triangular Fejér means of Walsh-Fourier series

Let x be an element of the unit interval $I := [0, 1)$. The $\mathbb{N} \ni n$th Walsh function is

$$\omega_n(x) := (-1)^{\sum_{k=0}^{\infty} n_k x_k}, \qquad n = \sum_{k=0}^{\infty} k_i 2^i, \; x = \sum_{k=0}^{\infty} \frac{x_i}{2^{i+1}}.$$

Define the triangular Walsh-Fourier partial sums $S_k^{\triangle} f$ of $f \in L^1(I^2)$ as:

$$S_k^{\triangle} f(x^1, x^2) := \sum_{i=0}^{k-1} \sum_{j=0}^{k-i-1} \hat{f}(i, j)\omega_i(x^1)\omega_j(x^2).$$

There is a sharp constrast between the trigonometric and the Walsh case. In 1987 Harris proved [2] for the Walsh system that if $1 \leq p < 2$, then there exists an $f \in L^p(I^2)$ such that $S_k^{\triangle} f$ diverges a.e. and in L^p norm. We talk about the investigation of the Fejér (or $(C, 1)$) means of triangular sums of two-dimensional Fourier series defined as:

$$\sigma_n^{\triangle} f := \frac{1}{n} \sum_{k=0}^{n-1} S_k^{\triangle} f.$$

For the trigonometric system Herriot proved [3] the a.e. (and norm) convergence $\sigma_n^{\triangle} f \to f$ $(f \in L^1)$. Till now with respect to the Walsh case there were only partial results. The main difficulty is that in the Walsh case there is no closed formula for the kernel functions. We also discuss the relation of this means and the Marcinkiewicz means of two-dimensional integrable functions.

References

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[2] D. Harris, *Almost everywhere divergence of multiple Walsh-Fourier series*, Proc. Amer. Math. Soc. **101** (1987), no. 4, 637–643.
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Roman Ger
(Silesian University)

On a problem of Navid Safaei

In the April 2016 issue of *The American Mathematical Monthly* (**123**, Problems and Solutions, p. 399) the following problem was proposed by Navid Safaei:

Let f be a function f from $\mathbb{R}$ into $[0,\infty)$ such that

$$f^2(x+y)+f^2(x-y)=2f^2(x)+2f^2(y)$$

for all x and y. Prove

$$f(x+y)\le f(x)+f(y)$$

for all x and y.

(Problem **11904**)

We have solved this problem in the affirmative. Several generalizations, including the general nonnegative solution of the functional equation in question, will also be presented.

Attila Gilányi
(University of Debrecen)

On computer assisted methods related to investigations of some inequalities and equations

There are several common points in the scientific activity of László Losonczi and Zsolt Páles. One of them is that they started to use computers (more precisely, computer algebra systems) in novel ways in the investigations of problems related to means. In connection with their results, we consider some computer assisted methods applied in the study of various classes of inequalities and equations.

László Horváth
(University of Pannonia)

Sharp Gronwall-Bellman type integral inequalities with delay

Various attempts have been made to give an upper bound for the solutions of the delayed version of the Gronwall-Bellman integral inequality, but the obtained estimations are not sharp. In this talk a new approach is presented to get sharp estimations for the nonnegative solutions of the considered delayed inequalities. The results are based on the idea of the generalized characteristic inequality. Our method gives sharp estimation, and therefore the results are more exact than the earlier ones.

Witold Jarczyk
(University of Zielona Góra)

Convex functions in abelian semigroup setting

(joint work with Zsolt Páles)

This is a short report on our recent research of convexity in abelian semigroups which is a continuation of our paper [2] dealing with convex subsets of abelian semigroups and, in a sense, of paper [1] devoted to convex functions defined on subsets of abelian groups.

Given an abelian semigroup S and an extended real-valued function $f\colon S \to [-\infty, +\infty]$ we define the *effective domain* and *epigraph* of f by

$$\mathrm{dom} f := \{x \in S \mid f(x) < +\infty\} \quad \text{and} \quad \mathrm{epi} f := \{(x, y) \in S \times [-\infty, +\infty) \mid f(x) \le t\},$$

respectively. For a fixed $n \in \mathbb{N}$ a function $f\colon S \to [-\infty, +\infty]$ is called *n-convex* and *n-konvex* if the epigraph of f is n-convex and n-konvex in the semigroup $S \times [-\infty, +\infty)$, that is if

$$n^{-1}([n]\mathrm{epi} f) \subset \mathrm{epi} f \;\text{ and }\; [n]\mathrm{epi} \subset n\,\mathrm{epi} f,$$

respectively.

In the talk we establish characterizations of both the convexity notions.

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[2] W. Jarczyk and Zs. Páles, *Convexity and a Stone-type theorem for convex sets in abelian semigroup setting*, Semigroup Forum **90** (2015), 207–219.

Tibor Kiss
(University of Debrecen)

Reducible means and reducible convexity properties

(joint work with Zsolt Páles)

It is a well-known fact concerning the *n-variable Jensen inequality* that it implies the *k-variable Jensen inequality* for all $k \in \{1, \ldots, n\}$.

Motivated by this phenomenon, the aim of talk is to investigate and answer the following question: Given two n-variable means $M\colon D^n \to X$ (where X is a topological vector space and $D \subseteq X$ is a convex set) and $N\colon J^n \to J$ (where $J \subseteq \mathbb{R}$ is an interval), how to define, for $k \in \{1, \ldots, n\}$, two k-variable means $M_k\colon D^k \to X$ and $N_k\colon J^k \to J$ which are naturally connected to M and N, furthermore, for any function $f\colon D \to J$, the n-variable inequality

$$f(M(x_1, \ldots, x_n)) \leq N(f(x_1), \ldots, f(x_n)), \qquad (x_1, \ldots, x_n \in D)$$

implies the k-variable inequality

$$f(M_k(x_1, \ldots, x_k)) \leq N_k(f(x_1), \ldots, f(x_k)), \qquad (x_1, \ldots, x_k \in D)?$$

Milica Klaričić Bakula
(University of Split)

Jensen-Steffensen inequality: old and new

(joint work with Marko Matić)

Let I be an interval in $\mathbb{R}$ and $f\colon I \to \mathbb{R}$ a convex function on I. If $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_m)$ is any m-tuple in I^m and $\boldsymbol{p} = (p_1, \ldots, p_m)$ any nonnegative m-tuple such that $\sum_{i=1}^{m} p_i > 0$, then the well known Jensen's inequality

$$f\left(\frac{1}{P_m}\sum_{i=1}^{m} p_i \xi_i\right) \leq \frac{1}{P_m}\sum_{i=1}^{m} p_i f(\xi_i) \tag{1}$$

holds, where $P_m = \sum_{i=1}^{m} p_i$.

It is known that the assumption $\boldsymbol{p}$ is a nonnegative m-tuple can be relaxed at the expense of more restrictions on the m-tuple $\boldsymbol{\xi}$. Namely, if $\boldsymbol{p}$ is a real m-tuple such that

$$0 \leq P_j \leq P_m, \qquad j = 1, \ldots, m; \quad P_m > 0, \tag{2}$$

where $P_j := \sum_{i=1}^{j} p_i$, then for any monotonic m-tuple $\boldsymbol{\xi}$ (increasing or decreasing) in I^m we get

$$\bar{\xi} = \frac{1}{P_m}\sum_{i=1}^{m} p_i \xi_i \in I,$$

and for any function f convex on I, (1) still holds.

Inequality (1) considered under conditions (2) is known as the Jensen-Steffensen inequality for convex functions. We can say that the Jensen-Steffensen inequality is "the ugly sister" of Jensen's inequality: not much admired and usually "not invited to the party". Our goal here is to show that "she" has many hidden beauties and that "she" can proudly walk hand in hand with her well known sister.

József Kolumbán
(Babeş-Bolyai University)

Is the Minty variational problem the dual of an optimization problem?

Let E be a normed vector-space, E^* its topological dual, $K \subset E$ a nonempty convex set and $T\colon E \to E^*$. We say that $\bar{u} \in K$ is a solution of the variational inequality if it satisfies

(VI) $$\langle T(\bar{u}), v - \bar{u} \rangle \geq 0,\ \forall v \in K.$$

We make the assumption that T is pseudomonotone, which means that for any $u, v \in E$, we have

$$\langle T(u), v - u \rangle \geq 0 \Rightarrow \langle T(v), v - u \rangle \geq 0.$$

Furthermore, we assume T is hemicontinuous, which means that

$$t \mapsto \langle T((1-t)u + tv), w \rangle,\ t \in [0, 1]$$

is continuous at 0 for every $u, v, w \in E$. It is well known that, if T is pseudomonotone and hemicontinuous, then (VI) is equivalent to the following problem, termed the dual (also called the Minty) variational inequality: find $\bar{u} \in K$ such that

(DVI) $$\langle T(v), \bar{u} - v \rangle \leq 0,\ \forall v \in K.$$

Our aim is to provide a general duality theory which justifies the term (DVI) for this problem, and at the same time links (VI)-(DVI) to the classical concept of duality in vector optimization.

Judit Kosztur
(University of Debrecen)

On conditionally polynomial functions

(joint work with Katarzyna Chmielewska and Attila Gilányi)

A function f defined on the real line $\mathbb{R}$ mapping into a linear space Y is called a polynomial function of degree n if it satisfies the functional equation

$$\Delta_y^{n+1} f(x) = 0$$

for all $x, y \in \mathbb{R}$, where n is a fixed non-negative integer. In this talk, we consider the situation, when the equation above is valid for some special elements $x, y \in \mathbb{R}$ only. Our investigations were motivated by some results on polynomial and linear functional equations presented in the papers [1], [2] [3] and [4], furthermore, by characterizations of polynomial functions via the Dinghas derivative (cf. [5] and [6]).

References

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László Losonczi
(University of Debrecen)

Multidiagonal matrices

Let n, k, l be fixed natural numbers with $1 \le k < l \le n$. We study pentadiagonal matrices $A = (a_{ij})$, $(i, j = 0, 1, \ldots, n)$ where

$$a_{ij} = \begin{cases} l_j & \text{if} \quad i = j + k, j = 0, \ldots, n - k, \\ L_j & \text{if} \quad i = j + l,\ j = 0, \ldots, n - l, \\ d_i & \text{if} \quad i = j = 0, \ldots, n, \\ r_i & \text{if} \quad j = i + k, i = 0, \ldots, n - k, \\ R_i & \text{if} \quad j = i + l,\ i = 0, \ldots, n - l, \\ 0 & \quad\ \ \text{otherwise.} \end{cases}$$

If $L_i = R_i = 0\,(i = 0, \ldots, n-l)$ then A becomes a tridiagonal matrix. We give a method to evaluate determinants of tridiagonal matrices and pentadiagonal matrices provided that $k+l \ge n+1$. Also discuss calculation of eigenvalues, eigenvectors, inverses of such matrices.

References

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Judit Makó
(University of Miskolc)

On strongly convex functions

(joint work with Attila Házy)

We would like to give connections between strong Jensen convexity and strong convexity type inequalities. The optimal Takagi type function of strong convexity will be examined too. Finally, we give some Hermite–Hadamard type inequalities in the case of strong convexity.

Gyula Maksa
(University of Debrecen)

Remarks on Hosszú's inequality

It is well-known that Hosszú's functional equation and the Jensen equation are equivalent under several conditions. Near 30 years ago, Z. Daróczy formulated the question: what can be said about the connection between the continuous solutions of the corresponding inequalities

$$f(x+y-xy)+f(xy)\leq f(x)+f(y) \qquad \text{and} \qquad f\left(\frac{x+y}{2}\right)\geq \frac{f(x)+f(y)}{2}.$$

In our talk, we present results connected with this question.

Janusz Matkowski
(University of Zielona Góra)

On some applications of a sandwich theorem

Let (Ω, Σ, μ) be measure space with at least two disjoint sets of finite and positive measure. By $S = S(\Omega, \Sigma, \mu)$ denote the linear space of all μ-integrable simple functions $x \colon \Omega \to \mathbb{R}$, and by $S_+ = S_+(\Omega, \Sigma, \mu)$ the set of all nonnegative $x \in S$ having support of positive measure.

For arbitrary bijection $\varphi\colon\ (0, \infty) \to (0, \infty)$, the functional $\mathbf{P}_\varphi \colon S_+ \to \mathbb{R}$ given by $\mathbf{P}_\varphi(x) := \varphi^{-1}\left(\int_\Omega \varphi \circ x d\mu\right)$ is well defined. By the converse of Minkowski's inequality theorem [2], under some modest conditions, if φ is nonpower and $\mathbf{P}_\varphi$ is subadditive in S_+, then either

(**I**) (Ω, Σ, μ) is generalized counting measure space, i.e. $\mu(\Sigma) \subset \{0\} \cup [1, \infty]$

or

(**II**) (Ω, Σ, μ) is a "degenerated" probability space, i.e. $\mu(\Sigma) \subset [0, 1] \cup \{\infty\}$.

In the case (i), applying a sandwich theorem [1], we characterize all continuous bijections φ such that the functional $\mathbf{P}_\varphi$ is subadditive.

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Flavia-Corina Mitroi-Symeonidis
(University of South-East Europe LUMINA)

Convexity and sandwich theorems

We review sandwich theorems from the theory of convex functions. We also provide a sandwich type theorem for convex set-valued functions as a counterpart of a known result in the context of the usual convexity.

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Jacek Mrowiec
(University of Bielsko-Biała)

On higher-order Wright-convex functions and Jensen-convex functions

In the talk we will show that for any positive integer number n the class of n-Wright–convex functions is the proper subclass of the class of n-Jensen–convex functions.

Gergő Nagy
(University of Debrecen)

Order preserving maps on structures of linear operators

(joint work with Lajos Molnár)

In this talk, some results concerning order preserving maps on sets of linear operators on normed spaces are presented. In the case of Hilbert space operators, the most basic order is the Löwner (or positive semidefinite) order between self-adjoint operators. We discuss the structure of the order automorphisms of the sets of positive, positive invertible and self-adjoint operators with respect to that order. Moreover, we consider other orderings, e.g. the chaotic order and the spectral order on sets of Hilbert space operators, and present the general form of corresponding order automorphisms. In the case of the spectral order, those automorphisms are defined on the spaces of so-called effects and self-adjoint operators on a complex Hilbert space. The corresponding results show that the forms of the latter automorphisms are very much different in the two and in the higher dimensional cases.

Noémi Nagy
(University of Miskolc)

Approximate convexity with respect to a subfield

(joint work with Zoltán Boros)

Let $\mathbb{F}$ be a subfield of $\mathbb{R}$ and X be a linear space over $\mathbb{F}$. Let $D \subseteq X$ be a nonempty $\mathbb{F}$-convex and $\mathbb{F}$-algebraically open set,

$$D^* := D - D := \{x - y \mid x, y \in D\},$$

and $\alpha \to D^* \to \mathbb{R}_+$ be a nonnegative even function. The function $f\colon D \to \mathbb{R}$ is called $(\alpha, \mathbb{F})$-convex if it satisfies the inequality

$$f\left(tx + (1-t)y\right) \leq tf(x) + (1-t)f(y) + t\alpha\left((1-t)(x-y)\right) + (1-t)\alpha\left(t(y-x)\right)$$

for all $x, y \in D$ and for all $t \in \mathbb{F} \cap [0, 1]$.

In this talk we show that the $(\alpha, \mathbb{F})$-convexity of a function f is equivalent to two other statements. Namely, a function $f\colon D \to \mathbb{R}$ is $(\alpha, \mathbb{F})$-convex if and only if the inequality

$$\frac{f(u) - f(u - sh) - \alpha(-sh)}{s} \leq \frac{f(u + rh) - f(u) + \alpha(rh)}{r}$$

is satisfied for all $r, s \in \mathbb{F}_+$, $u \in D$, $h \in X$ (where $u - sh, u + rh \in D$), or, equivalently, there exists a function $A\colon D \times X \to \mathbb{R}$ such that

$$f(u + rh) - f(u) \leq r\,A(u, h) - \alpha(rh)$$

for all $u \in D$, $r \in \mathbb{F}$, $h \in X$ (where $u + rh \in D$).

We also prove that under certain conditions for the function α, the mapping A described above can be written as

$$A(u, h) = \lim_{s \to 0,\ s \in \mathbb{F}_+} \frac{f(u + sh) - f(u)}{s}$$

for all $u \in D$, $h \in X$. Moreover, in this case the mapping $h \mapsto A(u, h)$ is positively $\mathbb{F}$-homogeneous and subadditive for every $u \in D$.

Kazimierz Nikodem
(University of Bielsko-Biala)

Some inequalities for strongly convex functions

Let $(X, \|\cdot\|)$ be a normed space, D be a convex subset of X and c be a positive constant. A function $f\colon D \to \mathbb{R}$ is called *strongly convex with modulus c* if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - ct(1-t)\|x-y\|^2$$

for all $x, y \in D$ and $t \in [0,1]$.

In the talk we present counterparts of some classical inequalities (in particular, the Jensen, converse Jensen and Hermite–Hadamard inequalities) for strongly convex functions. Conditions under which two functions can be separated by a strongly convex one and a Hyers-Ulam stability result for strongly convex functions are also given.

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Andrzej Olbryś
(Silesian University)

On a generalized version of a support theorem for Jensen-convex functions

Let D be a convex subset of a real linear space and let $f\colon D \to \mathbb{R}$, $\omega\colon D \times D \to \mathbb{R}$ be given functions. The function f is said to be an ω-*Jensen-convex* if the following inequality

$$f\Big(\frac{x+y}{2}\Big) \leq \frac{f(x)+f(y)}{2} + \omega(x,y)$$

holds for all $x, y \in D$. If $\omega \equiv 0$ then f is said to be *Jensen-convex*.

In our talk we give the necessary and sufficient conditions on the map ω under which for a given point $y \in D$ there exists an ω-*support function* i.e. the function $h_y : D \to \mathbb{R}$ satisfying the following three conditions:

(i) $h_y(y) = f(y)$,
(ii) $h_y(x) \leq f(x)$, $x \in D$,
(iii) $\omega(x,z) = h_y(\frac{x+z}{2}) - \frac{h_y(x)+h_y(z)}{2}$, $x, z \in D$.

Zsolt Páles
(University of Debrecen)

On cyclic inequalities with two-variable generators

Given a nonempty set X, a two-variable function $F : X^2 \to \mathbb{R}$, and a natural number $n \geq 2$, we consider cyclic inequalities of the form

$$(I_n) \qquad F(x_1, x_2) + \cdots + F(x_{n-1}, x_n) + F(x_n, x_1) \geq 0 \qquad (x_1, \ldots, x_n \in X).$$

Assuming that $F(x, x) = 0$ holds for all $x \in X$, one can easily see that, for all $n \geq 2$, (I_{n+1}) implies (I_n). On the other hand, in certain particular cases (I_2) also implies (I_n). In our main results, we give several necessary and sufficient conditions for (I_n) to be valid for all $n \geq 2$.

Paweł Pasteczka
(Pedagogical University)

Hardy property among invariant means

(joint work with Zsolt Páles)

The aim of this talk is to characterize so-called Hardy means, i.e., those means

$$M \colon \bigcup_{n=1}^{\infty} \mathbb{R}_+^n \to \mathbb{R}_+$$

that satisfy the inequality

$$\sum_{n=1}^{\infty} M(x_1, \dots, x_n) \le C \sum_{n=1}^{\infty} x_n \quad \text{for all sequence} \quad (x_n)_{n=1}^{\infty} \quad \text{of positive numbers}$$

in a family of invariant means. The smallest possible number C is used to be called a Hardy constant of M and denoted here by $H(M)$.

The main results stated that if $M_1, \dots, M_n$ are symmetric, homogeneous, increasing, Jensen concave and repetition invariant then the Hardy constant of their Gaussian product i.e. the unique mean satisfying

$$M_{\otimes}(v) = M_{\otimes}(M_1(v), \dots, M_n(v)) \quad \text{for all} \quad v$$

is the Gaussian product of respective Hardy constants. More precisely

$$H(M_{\otimes}) = M_{\otimes}(H(M_1), \dots, H(M_n)).$$

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Milan Petrík
(Czech University of Life Sciences Prague)

New sufficient condition for Mulholland inequality

An increasing bijection $f\colon [0,\infty) \to [0,\infty)$ is said to solve *Mulholland inequality* if

$$f^{-1}\big(f(x+u)+f(y+v)\big) \le f^{-1}\big(f(x)+f(y)\big) + f^{-1}\big(f(u)+f(v)\big)$$

for every $u, v, x, y \in [0,\infty)$. Mulholland has introduced this inequality as a generalization of Minkowski inequality and he has also provided a sufficient condition (we call it the *Mulholland's condition*) [3, 5]:

Theorem. *If both f and $\log \circ f \circ \exp$ are convex then f solves Mulholland inequality.*

We are interested in the open problem whether the Mulholland's condition is also necessary for f to solve Mulholland inequality and, furthermore, whether the set of functions solving Mulholland inequality is closed with respect to compositions.

In 1993, Matkowski and Świątkowski [4] have shown that a function that solves Mulholland inequality is necessarilly a convex homeomorphism. In 1984, Tardiff has provided a different sufficient condition [8]. In 1999, Schweizer posed a question [7] whether the Mulholland's and the Tardiff's condition are equivalent, or not. In 2002 Jarczyk and Matkowski have demonstrated [2] that the Tardiff's condition implies the Mulholland's one. An alternative proof has been also given by Baricz [1] in 2010.

Let $k \in [0,\infty)$. We say that f is *k-subscalable* if $\frac{f(a)}{f(b)} f\left(\frac{b}{a}x\right) \le f(x)$ for every $a, b \in (0,\infty)$ and $x \in [0,a]$ such that $b - a \ge k$. We present a new condition [6]:

Theorem. *If f is convex, k-subscalable, and linear on $[0,k]$ then it solves Mulholland inequality.*

This condition delimits a strictly larger set of functions than the Mulholland's one and, furthermore, it allows to show that the set of functions solving Mulholland inequality is not closed with respect to compositions.

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Michael Plum
(Karlsruhe Institute of Technology)

A computer-assisted existence proof for Emden's equation on an unbounded L-shaped domain

(joint work with Filomena Pacella and Dagmar Rütters)

We prove existence, non-degeneracy, and exponential decay at infinity of a non-trivial solution to Emden's equation $-\Delta u = |u|^3$ on an unbounded L-shaped domain, subject to Dirichlet boundary conditions. Besides the direct value of this result, we also regard this solution as a building block for solutions on expanding bounded domains with corners, to be established in future work. Our proof makes heavy use of computer assistance: Starting from a numerical approximate solution, we use a fixed-point argument to prove existence of a near-by exact solution. The eigenvalue bounds established in the course of this proof also imply non-degeneracy of the solution.

Bella Popovics
(University of Debrecen)

Convex structures induced by Chebyshev systems

(joint work with Mihály Bessenyei)

Recent developments has clarified that some tools of Convex Geometry are closely related to separation theorems obtained in the field of Functional Inequalities. This phenomenon has motivated investigating convex structures induced by Chebyshev systems. The present talk characterizes such a possible convex structure, completely describing its combinatorial invariants.

Teresa Rajba
(University of Bielsko–Biała)

On strongly delta-convex functions of higher order

The notion of a delta-convex function of n-th order is a particular case of the notion of a delta-convex mapping of n-th order between two normed linear spaces, which were introduced by R. Ger (1994) [1]. We define and study the notion of strong delta-convexity of n-th order that generalizes strong n-convexity studied by R. Ger and K. Nikodem (2011) [2] and T. Rajba (2011) [3]. We give a characterization of strong delta-convexity of n-th order (T. Rajba (2011) [4]), in general case, without any additional assumptions of differentiability of functions (which extend results given by T. Rajba (2011) [3] concerning strong n-convexity).

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Maciej Sablik

(Silesian University)

A characterization of polynomial functions

Let G and H be commutative groups. Then $SA^i(G;H)$ denotes the group of all i-additive, symmetric mappings from G^i into H for $i \geq 2$, while $SA^0(G;H)$ denotes the family of constant functions from G to H and $SA^1(G;H) = \mathrm{Hom}(G;H)$. We also denote by $\mathcal{I}$ the subset of $\mathrm{Hom}(G;G) \times \mathrm{Hom}(G;G)$ containing all pairs (α, β) for which $\mathrm{Ran}(\alpha) \subset \mathrm{Ran}(\beta)$. Furthermore, we adopt a convention that a sum over empty set of indices equals 0.

We present the following result.

Theorem. *Fix $N, M \in \mathbb{N} \cup \{0\}$, and let $I_{p,r}$, $0 \leq p + r \leq M$ be finite subsets of $\mathcal{I}$. Suppose further that H is uniquely divisible by $N!$ and let functions $\varphi_i \colon G \to SA^i(G;H)$, $i \in \{0, \dots, N\}$ and $\psi_{p,r,(\alpha,\beta)} : G \to SA^i(G;H)$, $(\alpha, \beta) \in I_{p,r}$, $0 \leq p+r \leq M$ satisfy*

$$\varphi_N(x)(y^N) + \sum_{i=0}^{N-1} \varphi_i(x)(y^i) = \sum_{p+r=0}^{M} \sum_{(\alpha,\beta)\in I_{p,r}} \psi_{p,r,(\alpha,\beta)} \left(\alpha(x) + \beta(y)\right)(x^p, y^r)$$

for every $x, y \in G$. Then φ_N is a polynomial function.

The above statement is a generalization of earlier results from [1, 2, 3, 4]. We also present examples of applications.

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Árpád Száz
(University of Debrecen)

Generalizations of a restricted stability theorem of Losonczi on Cauchy differences to generalized cocycles

By using partial and total generalizations of an asymptotic stability theorem of Bachyrycz, Páles and Piszczek [1] on Cauchy differences to semi-cocycles and pseudo-cocycles [4], we shall prove total and partial generalizations of a restricted stability theorem of Losonczi [2] on Cauchy differences to semi-cocycles and pseudo-cocycles. One of our main results is the following

Theorem. *If F is a symmetric semi-cocycle on an unbounded, commutative preseminormed group X to an arbitrary commutative preseminormed group Y, and S is a relation on X such that the intersection of the domain and the range of S is bounded, then*

$$\sup_{z \in X^2} \|F(z)\| \leq 5 \sup_{z \in S^c} \|F(z)\|, \qquad \textit{where} \qquad S^c = X^2 \setminus S.$$

An even subadditive function $\| \ \|$ of a group X to $\mathbb{R}$ is called a preseminorm on X, and the ordered pair $X(\| \ \|) = (X, \| \ \|)$ is called a preseminormed group.

It is no accident that the plausible assumption $\|0\| = 0$ is not needed. Namely, if $\| \ \|$ is a preseminorm on X such that $\|0\| \neq 0$, then by defining $\|x\|^\star = 0$ for $x = 0$, and $\|x\|^\star = \|x\|$ for $x \in X \setminus \{0\}$, we can obtain a new preseminorm $\| \ \|^\star$ on X such that $\|0\|^\star = 0$.

A function F of one group X to another Y is called a semi-cocycle on X to Y if

$$F(x, y) + F(u, y + v) + F(x + y, u + v) = F(x, u) + F(y, u + v) + F(x + u, y + v)$$

for all $x, y, u, v \in X$.

It is well-known that a Cauchy-difference is a cocycle. Moreover, in [3], we have proved that a cocycle is a semi-cocycle (pseudo-cocycle). Thus, in particular by letting F to be a Cauchy difference, Y to be a Banach space in the above theorem, and using an obvious generalization of the classical Hyers theorem, we can still obtain an immediate generalization of [2, Theorem 1] of Losonczi.

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László Székelyhidi
(University of Debrecen)

Spherical spectral synthesis

Spectral synthesis deals with the description of translation invariant function spaces. It turns out that – under some conditions – every function in such a space can be approximated by exponential polynomials belonging to the pointwise closure of the space. In this talk we exhibit how to extend the ideas of spectral synthesis from commutative to non-commutative groups using the theory of spherical functions and Gelfand pairs.

Éva Székelyné Radácsi
(University of Debrecen)

A new characterization of convexity with respect to Chebyshev systems

(joint work with Zsolt Páles)

The notion of nth order convexity in the sense of Popoviciu is defined via the nonnegativity of the $(n+1)$st order divided differences of a given real-valued function. In view of the well-known recursive formula for divided differences, the nonnegativity of $(n+1)$st order divided differences is equivalent to the monotonicity of the nth order divided differences which provides a characterization of nth order convexity.

The aim of this talk is to introduce the notion of higher-order divided differences in the context of convexity with respect to Chebyshev systems. Using a determinantal identity of Sylvester, we then establish a recursive formula for the generalized divided differences of consecutive order which enables us to obtain a new characterization of convexity with respect to Chebyshev systems.

Patricia Szokol
(University of Debrecen)

Transformations preserving norms of means of nonnegative functions

(joint work with Lajos Molnár)

Let X be a locally compact Hausdorff space. We denote by $C_0(X)$ the algebra of all complex valued continuous functions on X that vanish at infinity and the symbol $C_0(X)_+$ stands for the set of all functions in $C_0(X)$ whose values are nonnegative. The norm $\|.\|$ what we consider on $C_0(X)$ is the usual sup-norm. Any mean M on the nonnegative real numbers gives rise in an obvious way to a map that we call a mean on nonnegative functions and denote by the same symbol M. In fact, for all $x, y \in C_0(X)_+$ we define $M(x,y)(t) := M(x(t), y(t)), t \in X$.

Motivated by recent investigations on norm-additive and spectrally multiplicative maps on various spaces of functions, in our presentation we determine all bijective transformations between cones of nonnegative elements of certain algebras of continuous functions which preserve the sup-norm of a given mean of elements.

Csaba Vincze
(University of Debrecen)

On computable classes of equidistant sets: finite focal sets and equidistant functions

(joint work with A. Varga, M. Oláh, L. Fórián and S. Lőrinc)

Let K and L be two nonempty subsets in the Euclidean plane. The equidistant set of K and L is a set all of whose points have the same distance from K and L. Since the classical conics can be also given in this way [1] equidistant sets can be considered as a kind of their generalizations: K and L are called the focal sets. In general it is difficult to determine the points of an equidistant set because there are no simple formulas to compute the distance between a point and a set. As a simplification of the general problem we are going to investigate equidistant sets with special focal sets: equidistant sets with finite focal sets and equidistant functions. The main results give the characterization of the equidistant points in terms of computable constants and parametrizations. For the computer simulation of equidistant sets with final focal sets a MAPLE based algorithm will be also presented. Its importance is based on the continuity property of equidistant sets: if K and L are (not necessarily finite) nonempty disjoint compact subsets, $K_n \to K$ and $L_n \to L$ with respect to the Hausdorff metric then any bounded region of the equidistant set of K_n and L_n tends to the corresponding bounded region of the equidistant set of K and L; see Theorem 11 in [1]. Therefore we have a MAPLE based algorithm to approximate the equidistant points of K and L with the equidistant points of finite subsets in the plane. Such an approximation can be applied to the computer simulation as an alternative of the error estimation process for quasi-equidistant points suggested by [1] in subsection 4.2.

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Peter Volkmann
(Karlsruhe Institute of Technology)

A fixed point theorem for monotone operators in ordered Banach spaces

(joint work with Gerd Herzog)

Let E be a real Banach space ordered by a normal cone K. We assume $\Psi : K \to [0, \infty[$ to be such that for decreasing sequences (x_n) in K we have

$$\Psi(x_n) \to 0 \ (n \to \infty) \implies x_n \to \theta \ (n \to \infty)$$

(where θ denotes the zero element of E).

Theorem. *Consider* $a, b \in E,\ a \leq b,\ [a, b] = \{x \mid x \in E,\ a \leq x \leq b\}$ *and an increasing function* $g\colon [a, b] \to [a, b]$ *satisfying*

$$\Psi(g(y) - g(x)) \leq L\Psi(y - x) \qquad (x, y \in [a, b];\ x \leq y),$$

where $0 \leq L < 1$. *Then* g *has a unique fixed point* z, *and for each* $x \in [a, b]$ *we get* $g^{(n)}(x) \longrightarrow z\ (n \to \infty)$.

The result is published in KITopen (2016), 6 pp., DOI: 10.5445/IR/1000053743

Szymon Wąsowicz
(University of Bielsko-Biala)

Ohlin's Lemma as a tool in the theory of inequalities

(joint work with Jacek Mrowiec and Teresa Rajba)

A prominent rôle is played by Ohlin's Lemma in the theory of inequalities. Even though proved in 1969, Ohlin's Lemma is rather lesser-known to the wide mathematical community. Let $\mathcal{I}$ be a real interval and let X, Y be integrable, $\mathcal{I}$-valued random variables such that $\mathbb{E}X = \mathbb{E}Y$ and the cumulative distribution functions F_X, F_Y cross exactly once, that is to say, that there exists $t_0 \in \mathcal{I}$ such that

$$\big(F_X(t) - F_Y(t)\big)(t - t_0) \geqslant 0$$

for all $t \in \mathcal{I}$. Ohlin's Lemma states that under these hypotheses $\mathbb{E}f(X) \leqslant \mathbb{E}f(Y)$ for all continuous convex functions $f\colon \mathcal{I} \to \mathbb{R}$.

During the talk it will be demonstrated that this inequality may be employed to derive numuerous inequalities involving continuous convex functions in an elegant and unified way.

In 2014, during the *Conference on Ulam's Type Stability* held in Rytro (Poland), Ioan Raşa recalled his 25-years-old problem related to the preservation of convexity by the Bernstein–Schnabl operators. The Bernstein fundamental polynomials of degree n are given by the formulæ

$$b_{n,i}(x) = \binom{n}{i} x^i (1-x)^{n-i}, \qquad i = 0, 1, \dots, n.$$

Raşa asked to prove or disprove that

$$\sum_{i,j=0}^{n} \big(b_{n,i}(x)b_{n,j}(x) + b_{n,i}(y)b_{n,j}(y) - 2b_{n,i}(x)b_{n,j}(y)\big) f\left(\frac{i+j}{2n}\right) \geqslant 0$$

for each continuous convex function $f : [0,1] \to \mathbb{R}$ and for all $x, y \in [0,1]$.

In the main part of our lecture we shall explain why this inequality was possible to prove by using Ohlin's Lemma. The heuristics as well as a brief sketch of the proof will be given.

Alfred Witkowski

(University of Science and Technology)

New faces of the Hermite-Hadamard inequality

(joint work with Monika Nowicka)

Let $x_0, \dots, x_n$ be the vertices of a simplex $\Delta \subset \mathbb{R}^n$ and $N = \{0, 1, \dots, n\}$. For a set $K \subset N$ of cardinality $1 \le k \le n$ we denote by Δ_K the simplex with vertices $x_k,\ k \in K$. The simplices Δ_K and $\Delta_{N \setminus K}$ are called opposite faces of Δ. The point $b_K = k^{-1} \sum_{i \in K} x_i$ is called a barycenter of Δ_K. By $H(a, \lambda)$ we shall denote the homothety with center a and scale λ, i.e. $H(a, \lambda)(x) = a + \lambda(x - a)$.

We define also a set of simplices Δ^K as follows:

$$\Delta^K = H\left(b_{N \setminus K}, \frac{k}{n+1}\right)(\Delta_K).$$

All the simplices Δ^K contain the barycenter b of Δ.

If Σ is a k-dimensional simplex, then we shall denote by $\mathrm{Vol}(\Sigma)$ its k-dimensional volume. For an integrable function $f\colon \Sigma \to \mathbb{R}$ its average value over Σ is given by the formula

$$\mathrm{Avg}(f, \Sigma) = \frac{1}{\mathrm{Vol}(\Sigma)} \int_\Sigma f(x)dx.$$

If $f\colon \Delta \to \mathbb{R}$ is convex, then

Theorem 1 ([1]). *If $K \subset L$, then*

$$f(b) \le \mathrm{Avg}(f, \Delta^K) \le \mathrm{Avg}(f, \Delta^L) \le \mathrm{Avg}(f, \Delta).$$

Theorem 2 ([1]).

$$f(b) = \frac{1}{\binom{n+1}{1}} \sum_{\substack{K \subset N \\ |K|=1}} \mathrm{Avg}(f, \Delta^K) \le \frac{1}{\binom{n+1}{2}} \sum_{\substack{K \subset N \\ |K|=2}} \mathrm{Avg}(f, \Delta^K) \le \cdots$$

$$\cdots \le \frac{1}{\binom{n+1}{n}} \sum_{\substack{K \subset N \\ |K|=n}} \mathrm{Avg}(f, \Delta^K) \le \mathrm{Avg}(f, \Delta).$$

Theorem 3.

$$\mathrm{Avg}(f, \Delta) \le \frac{1}{\binom{n+1}{n}} \sum_{\substack{K \subset N \\ |K|=n}} \mathrm{Avg}(f, \Delta_K) \le \frac{1}{\binom{n+1}{n-1}} \sum_{\substack{K \subset N \\ |K|=n-1}} \mathrm{Avg}(f, \Delta_K) \le \cdots$$

$$\cdots \le \frac{1}{\binom{n+1}{1}} \sum_{\substack{K \subset N \\ |K|=1}} \mathrm{Avg}(f, \Delta^K) = \frac{f(x_0) + \cdots + f(x_n)}{n+1}.$$

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